Accurate Early Exercise Free Boundaries for American Puts

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Abstract
We present a numerical method for computing the free boundary problem for the American Put. A change of variable at each time step transforms the free boundary problem into a fixed one so that a mesh, that is refined near the "free boundary" is build once for all. We prove the accuracy of our numerical scheme with several examples.

1 Introduction

We consider an American put option with strike price K and maturity T. We note $r > 0$ the continuously compounded risk-free rate, $q \geq 0$ the continuous dividend yield and σ the local volatility. The option price V at time t and spot S satisfies the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} = rV \quad \text{for } T > t \geq 0, S > 0 \tag{1}$$

with the constraint

$$V(S, t) \geq \max(K - S, 0) \tag{2}$$

and the initial data

$$V(S, T) = \max(K - S, 0). \tag{3}$$

Here, r and q may depend on time, and σ may depend on time and spot.

This is a free boundary problem. It could be solved by the finite difference methods based on fixed spatial mesh. The traditional approach is to apply the American constraint explicitly. That is, Equation (1) is solved at each time step and the constraint is immediately applied to the solution. As it cannot guarantee the continuity of the Delta at the free boundary, this approach may cause instability and bad convergence when the time step is large.

Zvan, Forsyth and Vetzal (1998) (2000) have introduced the method of applying the constraints implicitly. That is, for each time step, one has to iterate on the grid points to find out the one below which the constraint (2) should be applied, and above which Equation (1) is satisfied. This grid point is the numerical approximation of the early exercise boundary.

Based on this idea, we have developed an efficient method for pricing American option, which we call **Coarseboundary**. **Coarseboundary** is optimized for grid adaptive refinement. As we shall see, **Coarseboundary** computes the American put option with high accuracy on theoretical value, delta and gamma, even with very small number of time steps and spot steps. Another advantage of the finite difference methods is that, with just one function call, we know the value of the option for all times and spot prices.

Once the space mesh is fixed, the difference between the numerical free boundary and the real exercise boundary is of the order of the size of the space discretization interval. With a standard space mesh of about 100 points, we cannot expect to capture the exercise boundary very accurately. As a consequence, when we plot gamma as a function of the underlying spot price, It may oscillate in the neighborhood of the free boundary.

Hence the idea: why not make a change of variables and let the exercise boundary be aligned with nodes of the computational grid? Based on this idea, we develop a new method which we call **Fineboundary**.

2 Fineboundary Method

For each t, we note $B(t)$ the early exercise boundary below which the American option must be exercised. Thus $B(t)$ satisfies:

$$\begin{cases} V(B(t),t) &= K - B(t) \\ \frac{\partial V}{\partial S}(B(t),t) &= -1 \end{cases} \quad (4)$$

If $B(t)$ were a known function, we could make the change of variables

$$\bar{S} = \frac{S}{B(t)}. \quad (5)$$

Equation (1) would become

$$\frac{\partial \bar{V}}{\partial t} + \frac{\sigma^2}{2}\bar{S}^2\frac{\partial^2 \bar{V}}{\partial \bar{S}^2} + \left(r - q - \frac{\dot{B}}{B}\right)\bar{S}\frac{\partial \bar{V}}{\partial \bar{S}} = r\bar{V} \quad \text{for } t > 0, \bar{S} > 1, \quad (6)$$

with both Dirichlet and Newman data on the fixed boundary $\bar{S} = 1$

$$\begin{cases} \bar{V}(1,t) &= K - B(t), \\ \frac{\partial \bar{V}}{\partial \bar{S}}(1,t) &= -B(t), \end{cases} \quad (7)$$

where $\dot{B}$ denotes $\frac{\partial B}{\partial t}$. The initial condition for B is $B(T) = min(K, Kr/q)$ (Kim (1990)).

The PDE (6) would be solved on a fixed $N \times N$ grid, with the Neumann data on $\bar{S} = 1$, where N is the number of steps. The grids being non uniform in general, we could further refine the space grid at $\bar{S} = 1$ and the time grid at $t = T$, in order to capture more accurately the rapidly changing solution.

As the free boundary is not known in advance, an iterative procedure will have to find the appropriate B and hence the change of variables so that the Dirichlet data is also satisfied.

In practice, the time discretization is based on a Crank-Nicholson scheme. If D_i and D_i^2 denote first and second discrete derivatives operators, defined by

$$D_i(V) = \frac{V_{i+1} - V_{i-1}}{S_{i+1} - Si - 1}$$

$$D_i^2(V) = \frac{2}{S_{i+1} - Si - 1}\left(\frac{V_{i+1} - V_i}{S_{i+1} - Si} - \frac{V_i - V_{i-1}}{S_i - Si - 1}\right),$$

Equation (6) is approximated by :

$$\frac{\bar{V}_i^{n+1} - \bar{V}_i^n}{\Delta t} + \frac{\sigma^2}{2}\bar{S}_i^2\frac{D_i^2(\bar{V}^{n+1}) + D_i^2(\bar{V}^n)}{2} + \left(r - q - \frac{2}{\Delta t}\frac{B^{n+1} - B^n}{B^{n+1} + B^n}\right)\bar{S}_i\frac{D_i(\bar{V}^{n+1}) + D_i(\bar{V}^n)}{2} = r\frac{\bar{V}_i^{n+1} + \bar{V}_i^n}{2}. \quad (8)$$

Given a guess B^n, previously computed B^{n+1}, and $\bar{V}^{n+1}$, Equation (8) gives $\bar{V}^n$. We write

$$\bar{V}^n = \text{solver}(B^n). \quad (9)$$

We introduce the function

$$f(B^n) = \bar{V}^n(1) - K + B^n = \text{solver}(B^n)(1) - K + B^n, \quad (10)$$

and we iteratively find B^n such that

$$f(B^n) = 0. \quad (11)$$

3 Numerical Examples

In this section, we present some numerical results of the **Fineboundary** method and compare them to those of **Coarseboundary** and Binomial method.

We have computed an American put option with $K = 100$, $T = 3$, $r = 0.06$, $q = 0.03$ and $\sigma = 0.1$.

Figure 1 shows the convergence of the computed exercise boundary at time $t = 0$ when we increase the number of steps N. We note that the approximated exercise boundary has already 4 exact digits with only 100 steps. Figure 2 shows that **Fineboundary** is far more accurate than the two other reference methods.

Figure 3 plots the approximation of early exercise boundary *vs*. time close to the maturity $T = 3$. Because of the singularity of the curve at maturity, $N = 100$ is not sufficient to compute a good approximation for

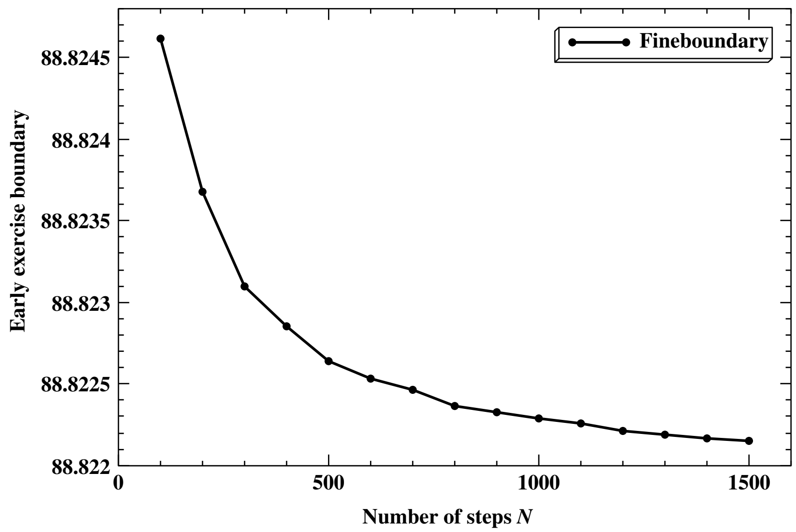

Figure 1: Early exercise boundary at t=0, with different number of steps

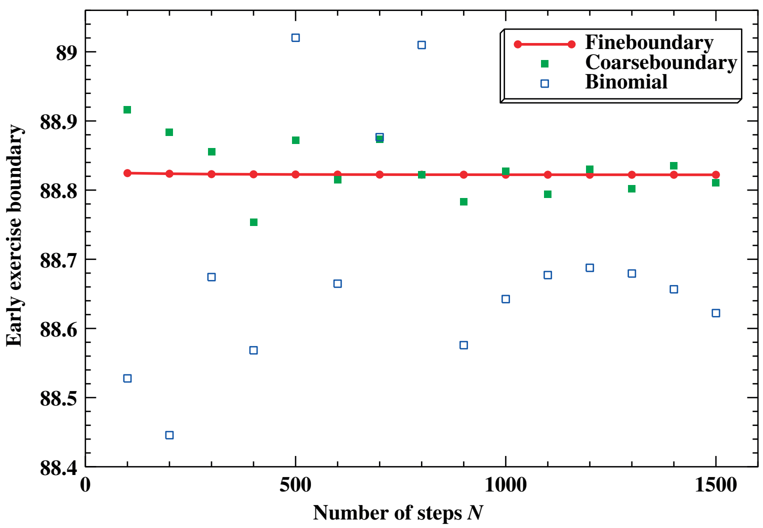

Figure 2: Early exercise boundary at t=0 computed by different methods

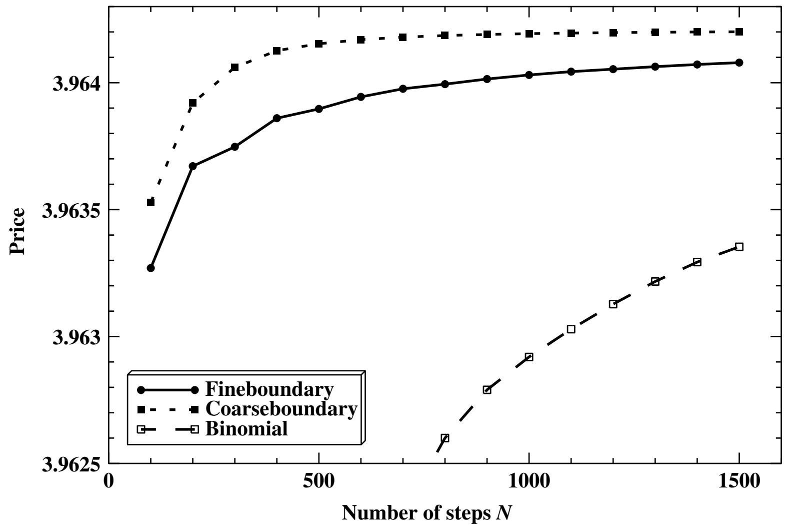

Figure 4: Convergence of the option price for S=100 computed using Fineboundary, Coarseboundary and Binomial method

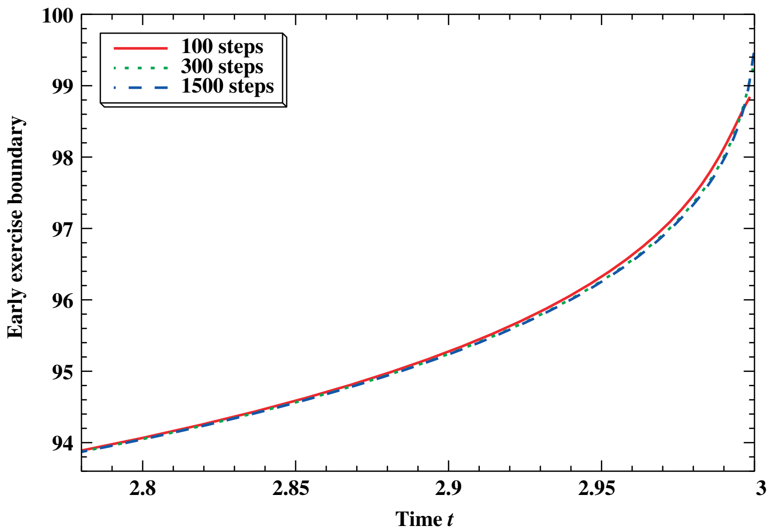

Figure 3: Early exercise boundary *vs.* time, computed with Fineboundary using 100, 300 and 1500 steps

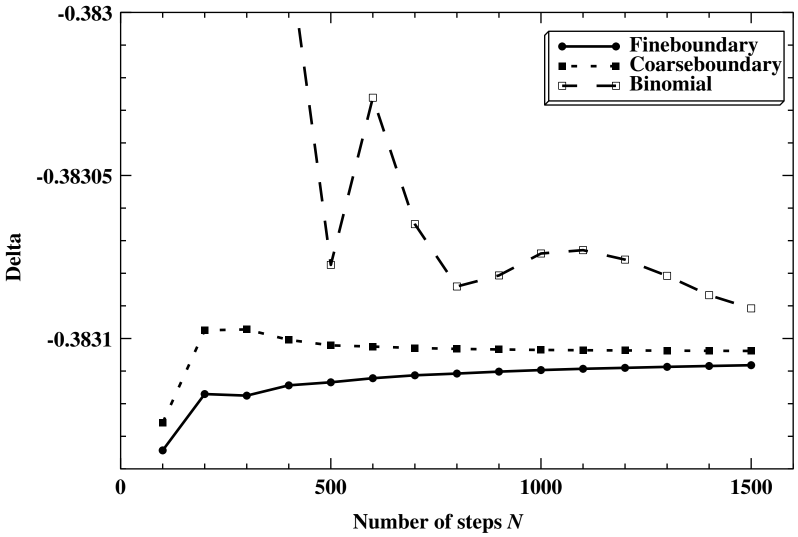

Figure 5: Convergence of option delta for S=100 computed using Fineboundary, Coarseboundary and Binomial method

$t > 2.85$. However, the exercise boundary computed with 300 steps fits already very well with that computed with 1500 steps.

We are also interested in how accurate the option price computed by **Fineboundary** method may be. Figure 4 plots the option price for $S = 100$ computed by **Fineboundary**, **Coarseboundary** and Binomial methods with number of steps varying from 100 to 1500. Figure 5 plots the delta value for $S = 100$ computed by these methods. Although **Fineboundary** gives a good result, **Coarseboundary** seems to do better!

Figure 6 plots the convergence of gamma computed for $S = 100$ by **Fineboundary** method and **Coarseboundary**.

But if we run **Coarseboundary** once and plot the gamma for different spot levels in the same mesh, we can see that the gamma oscillates badly in the neighbourhood of the exercise boundary, as shown in Figure 7 by the curve denoted by 'Coarseboundary 100 steps (one solve)'. The figure also plots the gamma value computed by **Fineboundary** method using 100 and 1500 steps. We see that the two curves are exactly the same.

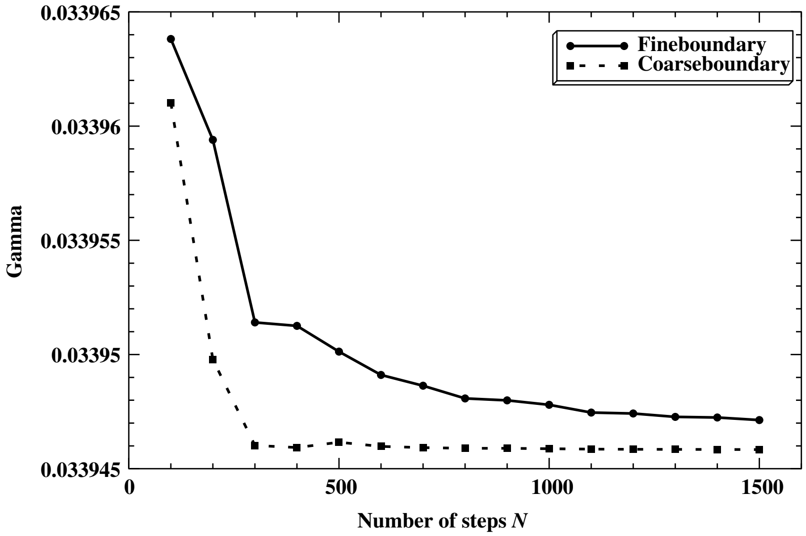

Figure 6: Convergence of Gamma for S=100 computed using Fineboundary and Coarseboundary

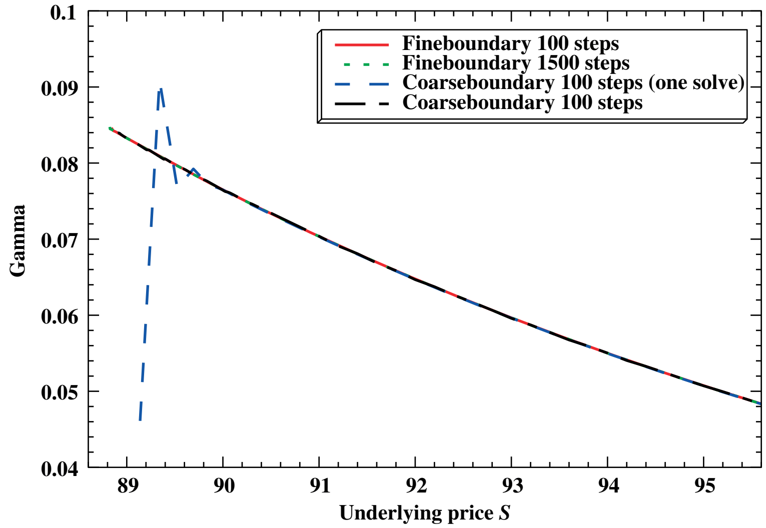

Figure 7: Gamma values computed using Fineboundary and Coarseboundary

However that does not mean that **Coarseboundary** fails to compute the gamma close to the exercise boundary. Indeed there are two ways that we can exploit a finite difference solver. Either the mesh is built once and for all for the given option, and values for the option and its greeks are interpolated off the grid for different spot levels and times (this is the method adopted in **Fineboundary**), or we can rebuild a mesh, centered and refined around the spot level, every time we want to compute values for that spot level. **Coarseboundary** is in fact based on the latter method. So we may as well plot gamma against spot level using **Coarseboundary**, only this time we would call the solver for every spot level and remesh for that spot level. The result is the curve denoted by 'Coarseboundary 100 steps' in figure 7.

We see that this curve fits perfectly the gamma computed by **Fineboundary** method. What happens in this case is that **Coarseboundary** remeshes and refines the mesh near the reference spot, hence near the free boundary, when it gets close to it. The free boundary is therefore captured more finely and gamma is more accurate.

Figure 8 shows the convergence of the gamma value for the spot 90, which is very close to the exercise boundary with two different methods. Although **Coarseboundary** yields a very accurate gamma for this spot, **Fineboundary** converges more smoothly with the increasing number of steps.

In conclusion, **Coarseboundary** performs very well, but **Fineboundary** gives an accurate gamma for all spots whereas we need to run **Coarseboundary** once for each spot, especially near the free boundary.

Table 1 shows the results for the option prices and deltas computed by the **Fineboundary** method with 100 steps. The results are compared to the "converged" values computed with a Binomial tree with a very large number of steps. We note that the absolute error of the price is around 0.001, and the deltas are very accurate. The results computed using 1500 steps are not displayed, since they are exactly the same as the Binomial benchmark values at the 3th decimal digit. We also show here the results calculated by the approach proposed by Lai, Yao, and AitSahlia (1999). The approach is based on the integral equation of Kim (1990). Even with a very small time refinement ($\delta = 10^{-4}$), their result does not match the quality of our solutions, both for the price and for the delta, as seen in Table 1.

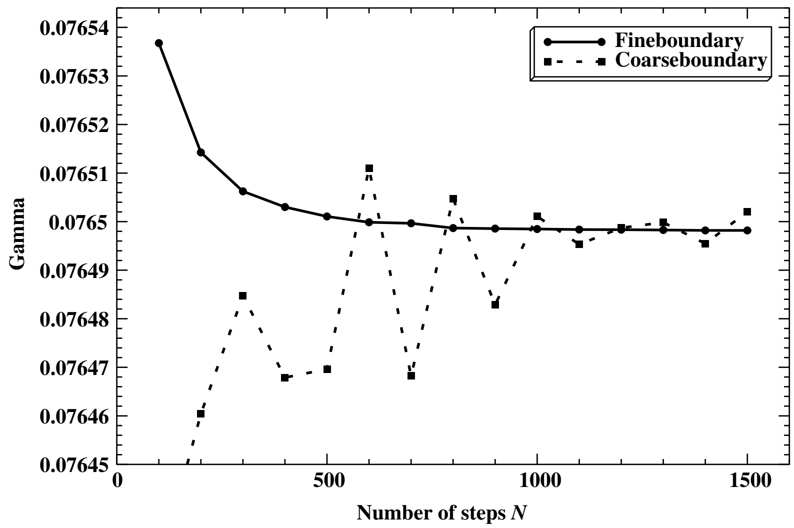

Figure 8: Convergence of Gamma on spot 90 computed using Fineboundary and Coarseboundary

TABLE 1. ACCURACY OF FINEBOUNDARY METHOD. CASE I: T=3, $\sigma=0.1$, $r=0.06$; CASE II: T=0.5, $\sigma=0.2$, $r=0.06$

	q	S	Price TrueVal	Price FB100	Price ALH	Delta TrueVal	Delta FB100	Delta ALH
Case I	.09	80	22.787	22.787	22.787	−0.754	−0.754	−0.754
		90	15.706	15.706	15.706	−0.655	−0.655	−0.655
		100	9.843	9.842	9.843	−0.511	−0.511	−0.511
		110	5.561	5.560	5.561	−0.346	−0.346	−0.346
		120	2.840	2.839	2.839	−0.204	−0.204	−0.204
	.06	80	20.000	20.000	20.000	−1.000	−1.000	−1.000
		90	11.593	11.591	11.598	−6.686	−6.686	−6.682
		100	6.087	6.085	6.082	−0.425	−0.425	−0.428
		110	2.870	2.867	2.870	−0.231	−0.231	−0.231
		120	1.221	1.220	1.221	−0.110	−0.110	−0.110
	.03	80	20.000	20.000	19.998	−1.000	−1.000	−1.000
		90	10.057	10.057	10.051	−0.905	−0.905	−0.904
		100	3.964	3.963	3.962	−0.383	−0.383	−0.383
		110	1.449	1.448	1.448	−0.152	−0.152	−0.152
		120	0.489	0.488	0.488	−0.055	−0.055	−0.055
	.00	80	20.000	20.000	19.992	−1.000	−1.000	−1.000
		90	10.000	10.000	9.984	−1.000	−1.000	−1.000
		100	2.723	2.723	2.727	−0.366	−0.366	−0.365
		110	0.706	0.706	0.705	−0.096	−0.096	−0.096
		120	0.178	0.178	0.178	−0.025	−0.025	−0.025
Case II	.09	80	20.802	20.802	20.803	−0.906	−0.906	−0.906
		90	12.422	12.421	12.422	−0.748	−0.748	−0.748
		100	6.183	6.183	6.183	−0.491	−0.491	−0.491
		110	2.535	2.534	2.534	−0.250	−0.250	−0.250
		120	0.865	0.865	0.865	−0.100	−0.100	−0.100
	.06	80	20.093	20.093	20.094	−0.949	−0.949	−0.949
		90	11.545	11.543	11.545	−0.742	−0.742	−0.742
		100	5.504	5.502	5.504	−0.462	−0.462	−0.462
		110	2.154	2.153	2.154	−0.223	−0.223	−0.223
		120	0.701	0.701	0.701	−0.085	−0.085	−0.085
	.03	80	20.000	20.000	20.000	−1.000	−1.000	−1.000
		90	10.953	10.951	10.953	−0.760	−0.760	−0.760
		100	4.961	4.959	4.961	−0.442	−0.442	−0.443
		110	1.843	1.842	1.843	−0.200	−0.200	−0.200
		120	0.570	0.570	0.570	−0.072	−0.072	−0.072
	.00	80	20.000	20.000	20.000	−1.000	−1.000	−1.000
		90	10.522	10.521	10.523	−0.794	−0.794	−0.794
		100	4.493	4.491	4.492	−0.427	−0.427	−0.427
		110	1.578	1.577	1.577	−0.180	−0.180	−0.180
		120	0.462	0.462	0.462	−0.061	−0.061	−0.061

The columns labeled TrueVal contain the values of the options obtained using a 20 000 step binomial lattice. The columns labeled FB100 contain the values computed by **Fineboundary** method using 100 steps. The columns labeled ALH contain the *hybrid* data reported by AitSahlia and Lai (2001).

4 Conclusion and Perspectives

In this work, we have introduced a new algorithm **Fineboundary**, to compute American puts. The results above show that the new algorithm is very accurate in capturing the early exercise boundary. But we have also shown that, as far as the accuracy of the option price, delta or gamma are concerned, it is not a crucial to adopt the new algorithm except when the region of interest is close to the exercise boundary. A method like **Coarseboundary** still produces a good gamma, which may be cahotically convergent in a very small error range. The problem with **Coarseboundary**, however, is that a new run is needed for every new reference spot. By contrast, **Fineboundary** produces an accurate gamma accurate for all spot levels at once! Moreover, we can exploit homogeneity, and use a single solve of **Fineboundary** to compute all puts of different strikes and given maturity. This is achieved by interpolation over different spot levels in the grid of **Fineboundary**. As the free boundary is perfectly captured in that grid, we can safely interpolate, even in its neighbourhood. This is not possible in **Coarseboundary** as shown in figure 7.

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