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A Truthful Generalization of Black–Scholes–Merton

BSM was never equipped to face an options market. The formula doesn't know what an options market is and even less so what the meaning of inverting the formula and implying volatility from the option market price could possibly be.

Two things come without an expiration date in finance and are supposed to last forever: the money account and the trading floor.

When Black, Scholes, and Merton (BSM) write down the self-financing dynamic trading strategy, $B + \Delta S$, with the purpose of replicating a contingent claim, both the amount of money B that they dynamically draw from the bank and the adjustable fraction of stock ΔS that they purchase with that money seem to spring from an inexhaustible source.

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From debt to equity

The bank and the stock exchange did have a beginning, though. While debt and credit are believed to be as old as the invention of writing, the origin of the stock exchange can be exactly retraced to the creation of the East India Company by the Dutch in 1602. A new form of finance was invented where, instead of loaning money to the entrepreneur or to the ship-owner and forcing him to liquidate his property in case of bankruptcy, the investor and the entrepreneur would jointly own stock in something called a "company." They would jointly bear the risk of its demise and, conversely, jointly participate in its potentially unlimited profits. Only when the first joint stock company was created did the bank start to accept shares of that company as collateral for loans, or to loan money in order that people may invest in those shares.

This mutual fungibility of credit and equity is the precursor of self-financing dynamic trading strategies, and as a consequence, shares indeed started being exchanged in the street. It is only later that a roof was added over the heads of the exchanging crowd in order to protect them from the rain, thus creating the place known as the "stock exchange." In the words of Niall Ferguson: "Company, bourse and bank provided the triangular foundation of a new kind of economy."[1]

Thus, the historical transformation of one mode of financing into another, or the historical conversion of debt into equity, or in other words the very mechanism lying at the heart of the convertible bond, is equivalent to the genesis of the market. This is the significance of the convertible bond and the reason why it will never go away.

The bank is perpetual, and trading is perpetual, and it is only in apocalyptic scenarios that their end can be imagined. The 1987 market crash almost terminated the bourse, as it was briefly envisaged that the Chicago Board Options Exchange would not open its doors the next day. And the 2008 financial crisis almost terminated the bank; the government had to bail it out. This is probably the reason why it has been said that "it is easier to imagine the end of the world than to imagine the end of capitalism."[2] It shouldn't come as a surprise, consequently, that debt structures which have been issued by banks after the financial crisis should include a bail-in clause on the one hand, and should be perpetual on the other hand. Pending the bailout event of the bank, which has now been internalized as a bail-in clause in the debt structure itself, the debt or the longevity of the bank can indeed become perpetual again.

Money and trading, although both perpetual, have different ways of enduring in time. Money as such is not supposed to move or to change, unless the end of the world takes place and your money becomes worthless, either by the failure of the bank or by the failure of money itself (hyperinflation). The hundred dollars that you have deposited in the bank (or loaned to the bank) are supposed to be returned to you whole, unless they are never returned. For this reason, the temporal process associated with credit is the Poisson process, or a jump process. In its most extreme case, the jump happens only once, the principal goes to zero, your money vanishes, and the world ends.

As soon as the stock was invented, however, and the money you had deposited in the bank became a dynamic account that you would use to finance your trading strategy in the stock, the temporal process changed its nature radically. Now motion became perpetual, when, in the case of debt, it is the anxiety of default that was perpetual. As soon as money left the bank for the street, a world of ventures and opportunities opened up. Indeed, the company whose stock you engaged in buying or selling actively had plans and projects and prospects reaching to all sorts of provinces of the world and extending in all sorts of time horizons, and no longer confined to your personal safe or your local bank. Speculation became the matter. Louis Bachelier could now write:

> *The influences that determine the movements of the Exchange are innumerable; past, current and even anticipated events that often have no obvious connection with its changes have repercussions for the price. Alongside the, as it were, natural variations, artificial causes also intervene: the Exchange reacts to itself and the current movement is a function not only of previous movements but also of the current state. The determination of these movements depends upon an infinite number of factors; it is thus impossible to hope for mathematical predictability.*[3]

From the infinity (literally!) of these factors influencing the stock price in ever tinier time intervals, Bachelier intuited Brownian motion, which he didn't call by this name. Vovk made this intuition rigorous, by deducing Brownian motion *in the absence of any prior probabilistic assumptions*, simply from the possibility of continuous dynamic trading in fractional size. Even the Wiener measure, which was named after the mathematician who rigorously constructed Brownian motion in a probabilistic setting, "emerges in a natural way in the continuous trading protocol," according to Vovk, therefore inverting the construction and putting trading before probability, action before longing and expectation, or once again, equity before credit.[4]

The equity-to-credit PDE

As the trading time horizon and the bank time horizon are potentially infinite (pending the event of collapse that abolishes the bank or the market or both), the only sensible measure becomes that of infinitesimal time. Continuous-time Brownian motion, reflecting continuous trading, is measured by instantaneous volatility, and the continuous-time Poisson process, reflecting the jump to bankruptcy that can happen anytime, is measured by instantaneous hazard rate. Volatility and hazard rate are the two key parameters of the equity-to-credit problem. In particular, they help unify the pricing problem of the convertible bond, which sits precisely at the juncture of equity and credit. They make explicit in the convertible bond price what really and causally affects it. By doing so, they lift all the mystery and all the vagueness that often attach to its pricing.

Given the hybrid nature of the convertible bond, it is not always clear, indeed, how the fixed-income component compares with the equity component or competes with it, and how valuation and risk should be balanced between the two. It is believed, for instance, that the fixed-income component should be discounted at the risky rate, and the equity component at the risk-free rate, without scientific justification either of this method of split discounting or indeed of the whole riskiness of the risky rate. Or it is said that risky companies can refinance at a lower cost by issuing convertible rather than straight debt. By that, it is meant they can get away with a lower coupon. This is again equivalent to saying that the convertible debt shouldn't be discounted at the full risky rate, and the implication is that the investor accepts to charge the company lower interest because of the advantage of becoming a shareholder should the company flourish. What we mean by "lower coupon" obviously depends on the value of the coupon that would normally attach to the straight bond as well as on the value of the option to convert the debt into equity. Various metrics were proposed to the fixed-income investor in order to regiment the convertible

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bonds under his customary criteria, such as the option-adjusted spread, or the modified duration, etc. However, these metrics depended on the maturity of the bond and ultimately relied on the methodology of discounting, which is characteristic of the fixed-income logic and may be questionable in the present case. Hence, they missed what the conversion into equity had precisely brought into the picture, namely the equity price dynamics and the necessity to analyze the outcomes, no longer globally and at a fixed horizon, but locally or over the next infinitesimal time interval.

Precisely, the notion of instantaneous hazard rate provides the causal explanation of the existence of credit spreads in the unified framework of risk-neutral pricing. Let it be noted, in this regard, that the BSM breakthrough didn't just attach to option pricing and its liberation from risk preferences. For, its later rigorization with martingales provided the general arbitrage-free framework in which to price anything whose payoff ultimately depended on some underlying stochastic dynamics. For instance, when the underlying stock price follows geometric Brownian motion, in the risk-neutral probability:

$$\frac{dS}{S} = rdt + \sigma dZ$$

a derivative such as a call option, whose arbitrage-free value is V, admits the following differential:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}\right)dt + \sigma S\frac{\partial V}{\partial S}dZ$$

Taking expectations, and requiring that the call option value process should be a discounted martingale:

$$E(dV) = rVdt$$

yields the following equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

after noticing that the expectation of the dZ term is zero. This is the BSM equation.

The equity price dynamics of a given issuer may not affect the value of its straight bonds, however the underlying dynamics now generalize, in his case, to include the possible jump to default:

$$\frac{dS}{S} = (r+\lambda)dt + \sigma dZ - dN$$

where N is the counting process of an independent Poisson process triggering default with intensity λ, also known as the hazard rate, and where the assumption is that the stock process undergoes a jump of –100 percent when this happens.

Both equity and debt become contingent upon this generalized dynamics, as the jump to default now drives both to their recovery value. In exactly the same way that option value, following BSM, is the discounted mathematical expectation of its payoff under the risk-neutral probability, the bond value is the discounted mathematical expectation of its payoff under the same probability, where its payoff is now contingent on the state of default or no default, and is equal to the recovery value V_d in case of default. The differential of the bond value writes:

$$dV = \frac{\partial V}{\partial t}dt + (V_d - V)dN$$

Taking expectations and applying the discounted martingale condition yields the following equation, after observing that $E(dN) = \lambda\, dt$ and letting $V_d = R_d V$:

$$\left[\frac{\partial V}{\partial t} + \lambda V(R_d - 1)\right]dt = rVdt$$

The straight bond is a trivial derivative, in the sense that it is not even a derivative on the issuer's equity. In this case, its value obeys the ordinary differential equation above, whose integration yields exactly the familiar continuous compounding expression, in which the instantaneous interest rate is augmented by the hazard rate multiplied by one minus the recovery rate R_d, or in other words, augmented by what the fixed-income analysis recognizes as the credit spread.

The convertible bond is a hybrid derivative, combining equity exposure and credit exposure. However, it should be treated as a single security, and no longer be decomposed into two parts. Precisely, the differential of its value encapsulates all the changes that might impact it in the next infinitesimal time interval, caused either by the continuous motion of the underlying equity or by the jump to recovery value in case of default:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+\lambda)S\frac{\partial V}{\partial S}\right)dt + \sigma S\frac{\partial V}{\partial S}dZ + (V_d - V)dN$$

This expression is all we need to justify the hybrid nature of the convertible bond. As a matter of fact, the value of both debt and equity, as well as of any general contingent claim, becomes the solution of the one and only unified partial differential equation (PDE), involving the instantaneous volatility and hazard rate as parameters. We derive it after taking expectations in the expression above and applying the martingale condition:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+\lambda)S\frac{\partial V}{\partial S} = (r+\lambda)V - \lambda V_d$$

Changes of value of the convertible are constrained by the boundary conditions. For instance, in periods when the bond is callable, its value should be kept below the early redemption value (or call strike), simply because the issuer optimally exercises the right to call. In periods when it is convertible, its value should be kept above the conversion value (or parity), because the holder has the option to convert it into equity and therefore to enforce the conversion value at least. A hierarchy must be respected when the rights enter into conflict. The holder keeps the right to convert in case of a call notice by the issuer.

As is apparent from the martingale approach, the current price of any security is equal to its discounted mathematical expectation. Valuation therefore proceeds backwards, starting from the initial conditions, or terminal payoff. This allows, for instance, adding the coupons to the solution we are evolving in the PDE every time the backward valuation procedure encounters a coupon date.

Convertible bond vs. CoCo bond

When dealing with simple equity options or when dealing with straight bonds, analytical formulas are sufficient for the purpose of valuation: the Black–Scholes formula, in the first case, or the traditional discounting formulas, in the second case. However, the valuation of the convertible bond and the way equity risk (volatility) and credit risk (hazard rate) intermingle in the corresponding PDE require numerical techniques that can only be handled by computer programs. This is what leads Michael Youngworth, Head of Global Convertible Strategy at BofA Securities, to note, in an article published by *Barron's* and dedicated to the boom in convertible bond issuance during the coronavirus crisis, that "companies with volatile stocks can get particularly

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low yields in the convertible market because they're favoured by the computer models used by many institutional buyers."[5] By computer models, Youngworth means the PDE we have described above, which is nothing other than the proper way of valuing convertible bonds.

The coronavirus pandemic has caused the stock market to fall sharply, reaching its lowest level in mid-March, and has put airlines as well as travel companies under severe stress. This has meant a sharp increase in their equity price volatility, as well as in their credit spread. Precisely, the point of the article is that the first has compensated the second at the time of issuance of convertible bonds, with the result of refinancing at lower cost. When applying the unified PDE, it should come as no surprise, then, that (citing *Barron's*) a stressed company, such as Carnival Corp., could secure an issue of $2 billion of convertible bonds with a 5.75 percent coupon, while at the same time failing to raise regular debt for less than 11.5 percent yield. As a matter of fact, the convertible bond and the straight bond are solutions of the same PDE, in which the volatility and the hazard rate parameters depend on the same common issuer and not on the specific bond. Only their initial conditions differ. This means that simply switching the payoff of the bond at maturity, between straight redemption of the principal (straight bond) and the greater of the principal and the conversion value (convertible bond), is sufficient to explain the difference in size between the coupons.

The hazard rate of Carnival Corp., or its instantaneous risk-neutral probability of default conditional on default not having occurred earlier, was quite high at the moment of issue, April 2, 2020. This explains the huge spread and consequently the large coupon (11.5 percent) that investors have asked for the straight debt. For reference, the 2Y credit default swap (CDS) was trading above 2000 bps that day. To complete the valuation of the convertible bond on the same day, and apart from the stock price which had reached its lowest level ($8), all one needs is an estimation of the volatility of that stock price. Conversely, one can imply its volatility from the known value of the convertible bond. We independently imply the hazard rate, on the issue date, from the known issue price of the 11.5 percent straight bond (which, in the case of Carnival Corp., was issued almost simultaneously with the convertible bond). Implied volatility is then the number which, when plugged into the PDE above, yields the known issue price of the 5.75 percent convertible bond. This amounts to implying volatility from the price of the option to convert $1000 worth of debt into 100 underlying shares (as dictated by the conversion ratio), which is a call option with a strike price of $10. We find the equivalent BSM implied volatility to be equal to 60 percent on the issue day.

The 2008 crisis was a financial crisis and its main actors were the banks. They were both the victims and the offenders. It was a credit crisis, fueled by the defaults of the mortgage owners, then propagated through the collateralized debt obligations and the very fragile model of default correlation they were predicated on, and finally concluded with the collapse of one major bank and the bailout of the others, which were deemed "too big to fail." The two main financial innovations that emerged in the aftermath were bitcoin and the contingent-convertible (CoCo) bond. The first expressed a total loss of faith in any centralized banking system or even monetary authority, and the second converted the banking moral hazard and bailout risk (which is all too often its correlate) into a bail-in clause. Equity once again came to the rescue of debt, only negatively so. Indeed, the investor who buys a CoCo bond issued by a bank does not have the option to convert it into shares. Rather, conversion into equity is forced upon him at the worst moment, precisely when the deteriorating financial health of the bank triggers its bail-in event or the writing down of its debt. The default chain reaction is thought to be defused in this way and the curse of credit to be removed from the system. Precisely, canceling the coupons in a non-cumulative way, and partially or totally writing down the principal, are not considered as default events. This, we may say, is the whole purpose of the exercise.

Some have argued that conversion of debt into equity should be total and global among mankind, and that even students should no longer present their credit to the bank in exchange for the money it is loaning them, but rather their "equity." By that it is meant they would offer to the bank financial participation in all their future life achievements or conversely, failures. This is all too good;

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however, there is one form of credit which by definition cannot disappear, and that is your own bank account. Even in an economy where only equity would be exchanged, and supposing the bitcoin network has not yet replaced all the banks, there would still exist the credit risk of the bank itself. Probably for the reason that the bank will always be sitting on the other side of money, the conversion of its debt (in theory perpetual) into equity will always have the form of a reverse convertible bond, or a CoCo bond.

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The 2020 crisis was a "physical" crisis, by comparison (not too sure we should speak in the past, here). It concerned the productive forces of the economy, and not the financial system. If anything, the banks were the least concerned by it. Of course, the stock market collapsed, and credit spreads widened, and bankruptcies loomed large, but by far the most remarkable and idiosyncratic effect of the crisis was the rebound of the stock market. With central banks stepping up ultimately to buy all kinds and all grades of debt, and with government bond yields being equal to zero, credit was no longer the issue and the only concern left among investors was the fear of missing out (FOMO). This is the fear of missing the stock-buying opportunities that the decline in their prices had produced and the fear of being excluded from the subsequent trading activity. To have the effervescence of the stock market solely in mind, and to privilege the trading action over anything else, are characteristic of the movement of conversion that we have described at the beginning. We said that the convertible bond was emblematic of the movement from the frozen mirror of the bond (debtor and creditor petrified by each other's image, awaiting the earlier of redemption or default) to the perpetual motion of the stock (speculation). When the stock price falls spectacularly, thus creating a spectacular rebound opportunity and sending volatility to the roof as a net result, and when, concomitantly, default risk increases because it is cruise ships we are talking about, at a time when sea travel looks far more hazardous than during the days of the Dutch East India Company, this designates the convertible bond as the perfect investment vehicle. 2020 is the year of the convertible bond, we may say.

Convertible bond arbitrage

The stock price of Carnival Corp. has more than doubled since its lowest, and the day the convertible bond was issued. This swing has certainly more than profited the convertible bond arbitrageur who had bought the embedded equity option for 60 percent implied volatility. How profits can be made by volatility arbitrage is by noting that the real volatility of the stock price during the same period was setting records at 250 percent, and by using a computer model like the one Youngworth is mentioning, which, once again, is nothing but the numerical implementation of the PDE above. Trading is dynamic and the PDE paints a dynamic picture by essence. It is the exact mathematical expression of the formidable transition from the world of fixed income and the corresponding discounting methodologies to the world of equity price dynamics and the dynamic hedging strategies first introduced by BSM. As a matter of fact, it is only half of the story to say that debt was converted into equity (for the first time by the Dutch in 1602), as a result of which the stock market was born and dynamic trading was invented. For the complete story is the invention of the derivative. The expression $B + \Delta S$, in which B (for bond or bank) is now geared to purchasing a variable amount Δ of S (for stock), this expression now involving the bank account and the stock market in the new economy Niall Ferguson was talking about, is actually nothing other than the self-financing dynamic trading strategy to replicate the payoff of a derivative. It is the formula for *writing* the derivative. An alternative to the martingale approach, and to setting the price of the derivative equal to its discounted expectation under the risk-neutral probability, is the dynamic hedging approach, first introduced by BSM in their 1973 paper. While both approaches lead to the PDE above, someone like Neftci will specifically call the latter the "PDE approach."[6] Probably the martingale approach, and the discounting and expectation procedure essentially characterizing it, were still remindful, in his eyes, of the fixed-income logic which, as its very name indicates, radically differs from the logic of dynamic trading in fractional size.

The convertible bond was emblematic of the movement from the frozen mirror of the bond (debtor and creditor petrified by each other's image, awaiting the earlier of redemption or default) to the perpetual motion of the stock (speculation)

For the buyer of an option, volatility arbitrage consists in purchasing it at a lower price than what the real volatility would dictate when plugged in the

PDE above. For instance, buying the Carnival Corp. convertible bond (and the embedded equity option) at the moment it was issued implied a future underlying stock price volatility of 60 percent, when the 30 days historical volatility, which by construction doesn't record a possible jump to default, was in fact equal to 250 percent. The fair price of the convertible bond, in these conditions, should have been equal to $200. To realize the potential profit, the volatility arbitrageur need not find a party to whom to sell back the convertible bond at this astronomical price. He only needs a counterparty to buy from him the *synthetic* convertible bond (i.e. someone to whom to sell the dynamic replicating strategy dictated by the computer program above, with the 250 percent volatility number plugged into it). This counterparty is simply the underlying stock market itself, in which the arbitrageur executes his dynamic replication. To complete his hedging, he can short the straight bond or indeed buy a CDS in a dynamic ratio that is also provided by the computer program or the PDE. For not only does the CDS, if purchased, protect him against a default event that might wipe out the fixed-income component of the convertible; it can also help him finance the extra option premium resulting from default risk, if sold. The net ratio is not trivial, as a result.

How the PDE demystifies everything and resolves the question of proper discounting of the convertible bond, or the question of the size of the coupon to charge its issuer, is thus by referring everything back to the market. The market consensus over the creditworthiness of Carnival Corp. in the middle

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of the coronavirus crisis is a refinancing cost of 11.5 percent annually. From this we infer the hazard rate of the company (i.e. its instantaneous risk-neutral probability of default, conditional on default not having occurred earlier), and all we subsequently need in order to discover the price of the convertible bond is the independent market pricing of the equity option. The universality of the equity-to-credit PDE is such that we can go back and forth between all these different instruments, all supposed to be priced by one and the same arbitrage-free market. Thus the market is the only rule, and while the PDE, as a consequence, is doomed to provide only relative and no fundamental valuation, the trading dynamics in which it is immersed potentially turns any instrument of the equity-to-credit universe into a hedging instrument of any other. The BSM paradigm, which is the child of self-financing dynamic trading, which is the child of the conversion of debt into equity, does not stop at the BSM equation, in our eyes. The equity-to-credit PDE is its first extension, certainly an indispensable one as it brings together the two continuous-time processes that we said the temporality of the market was made of, namely Brownian motion and the default Poisson process. We see the BSM paradigm as *the full opening to the market both of valuation and of risk analysis*, and while trading was supposed to be perpetual, now we can see it must be universal too. Hence, we should expect the next extension to introduce stochastic hazard rate and stochastic Brownian volatility, as this is what trading the CDS and the equity option (and potentially using them as hedging instruments) means. In contrast with the PDEs of the physical sciences, a PDE such as the one written above is never final; it is just one stage, or one regime, of trading, inside which all the others are virtually implicated. We will see how.

Indefinite time vs. finite time

The convertible bond sits at the hinge between credit and equity, and of their two radically different temporalities; for this reason we may call it a *critical* instrument, most suitable for the moment of crisis, when the company is on the brink between total collapse, on one side, and total rebound and recovery, on the other. The hazard rate and volatility are at their highest at that moment, and fate had better be sealed quickly indeed. Either the company files for bankruptcy and disappears, or its debt is converted into equity and it joins the endless activity of the stock market. As a consequence, the maturity of the convertible bond is usually short. Conversion is often forced as soon as the embedded option becomes 130 percent in the money. The bank, by contrast, lives a different form of crisis and experiences time differently. It is actually torn between two forms of temporality. On the one hand, it is the bank, as such perpetually indebted to its depositors and in charge of keeping the wheel of trading perpetually moving. On the other hand, it itself endures credit; it lives in finite time in which collapse is a constant threat, and for this reason it has to show continual signs of its good financial health. At least, this has been the case since the last financial crisis.

This unusual temporal duplicity (infinite because it is a bank, finite because it has credit) introduces the periodic call schedule inside the perpetual structure of debt. Although the CoCo bonds that banks have been issuing since 2008 have no finite maturity, the early redemption option (call), usually occurring after 5 years, and then periodically every 5 years, is perceived as a moral *obligation* to redeem the principal rather than just the right to do so. It has more or less been a gentleman's agreement between the bank and the investor that the bank would refinance at the next call date and the investor would get his money back. This break in time and the necessity to look in time when we were supposed never to look (perpetuity) entail, when we think about it for a second, *stochastic credit*. The perpetual maturity of the bond issued by the bank reflected the notion that the end of the world should be deferred perpetually. The end of the world was certainly a possibility; however, its probability had no measure and no scale. It was something absolute and wasn't a measurable possibility. Now the call schedule and the finite time that is thus introduced within the infinite time horizon of the bank, and before the end of time, suddenly cause a *quantification* of probability. They introduce a scale, and therefore a rescaling. As we now have to go check the credit of the

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bank regularly, when we were structurally supposed never to do so, this credit becomes measurable. We quantify it, we desacralize it, we make it relative, and this means that we make it stochastic in time. Finite time, when we think of it, is the same thing as stochastic credit.

If the credit of the bank were never to change, a perpetual coupon bond issued at par would be the same as a perpetual coupon bond with a periodic call option struck at par. There would be no difference between a situation in which the bank periodically redeems its debt, only to refinance it immediately, and a situation in which its debt is perpetual. But there is a difference. Finite time has broken into indefinite and indifferent time. And now the only way to handle a perpetual bond with a periodic call schedule in the face of stochastic credit is to assume that the stochastic process is a mean-reverting Markov process. The unique and steady state in which nothing ever changes, in which time is indifferent and a perpetual bond is the same as a periodically callable perpetual bond, generalizes into a Markov steady state whose simplest instance is a discrete two-state process, one in which the issuer calls the bond at the next call date, because his credit is good and he can refinance at better market conditions, and one in which his credit is bad, the redemption is deferred, and the bond is extended.

Credit and volatility regimes

Two things are perpetual in finance, we said, the bank account and the trading floor, B and ΔS. Debt was converted into equity, the stock market came into being, and the money account, B, now geared to dynamic trading in fractional size, ΔS, produced $B + \Delta S$, or the replication formula of the derivative. We saw how finite time broke into this picture and imposed a quantification of the credit of the bank in at least two stochastic regimes, one good and one bad, so we now wonder how finite time will also break into the other timeless and still unquantified variable, namely volatility. Surely, the end of the bank account or the end of the world could only, at first, happen outside of probability, that is to say, immeasurably, because it is the end of the world we were talking about, and, surely, the crisis of the bank subsequently imposed finite time upon this indefinite time and made us look when we were not supposed to look, thus quantifying the probability of default of the bank and turning it stochastic by the same token. But what could the end of the world of volatility be? What could be a similar movement whereby finite time, or the fact that we start looking when we were not supposed to, would impose a similar quantification of volatility, eventually leading to stochastic volatility?

The bank B was sitting on the left-hand side of the market activity $B + \Delta S$ (it is sitting on the other side of money), and this was the reason, we said, why its impossible or immeasurable bankruptcy, or the immeasurable end of its world, had to engender the call schedule, the superposition of the two temporalities and stochastic credit, as a result. But what would be the symmetrical fate of the entity sitting on the right-hand side of $B + \Delta S$? What would be the symmetrical end of the world, engendering a symmetrical superposition of temporalities? What would be the rescheduling and rescaling, no longer of the indefinite time of credit, but of the indefinite time of trading and volatility? What would be the result of metamorphosis of the regimes of credit through the conversion?

The answer must lie in what $B + \Delta S$ is achieving as it moves from left to right, in other words, it must lie in the derivative that is thus being replicated and written. The answer must be that the newly written derivative will bring about the quantification of the still unquantified volatility, and subsequently the stochastic regimes of volatility. If volatility should ever have been indefinite and constant, it would have had to be so in the world, not in time, and writing the derivative would have to be the end of that world. Stochastic changes of volatility would have, then, to occur, not in time, as every econometrician thinks, but through changing the world, *through the trading of the derivative*.

Volatility is constant in BSM because it is a concept and not a numerical magnitude. Insofar as the fateful mirror of debt leads to the eventful speculation of Bachelier, insofar as Vovk deduces Brownian motion from trading in fractional size and without probability, the perpetual motion and the randomness at any scale that are thus deduced are and remain unquantified concepts (that is to say, they are qualitative). They are just the expression of the *concept* of perpetual trading that has just replaced the perpetual bank account. Volatility is, for now, just the *idea* that the stock price has to be volatile because of the efficient market hypothesis. This concept of volatility is indefinite (in the sense of lasting forever and not being numerically defined); it reaches until the

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end of times. When BSM say: "Let volatility be σ," they have no idea of breaking volatility in time or in space, so to make it stochastic eventually, or even to measure it. They don't even bother to say how volatility is numerically inferred or estimated. This, of course, *is not* an indication that such a measurement would soon have to follow as a complement to their theory or concept. Simply, volatility is a constant symbol, not a number. Quantity hasn't yet broken inside quality. As a matter of fact, BSM handle one option at a time, of a given strike and maturity. That the strike and maturity can vary, and the option greeks display different profiles as a result, does not mean that more than one option is considered. The strike and maturity are parameters in the BSM equation. We "multiply" the number of options only by repeating or reconfiguring the one option. We do not consider a *multiplicity of options sharing a certain reality*, for instance the reality of the market, which will generate the smile problem. To consider multiple options in a reality (i.e. outside the formula in which the strike and the maturity are only parameters that repeat and reconfigure the one option) is to consider them in the market – what else? – and therefore to consider that they can be independently traded. The only reality of considering multiple options would thus have to be the smile problem, no? Volatility is a parameter too, in BSM. We repeat and reconfigure the option, thus changing its strike and maturity, but we also repeat and reconfigure volatility. It could be different too, without any relation or shared reality with the previous instance. The only situation in which the two options are given simultaneously, if it is to make any difference at all with respect to the formal situation in which they are never given simultaneously but are only repetitions one of another, must be a situation in which the same volatility *does not* simultaneously explain their two values. And this can only mean that they trade independently of each other and that this value is a market price. However, BSM was never equipped to face an options market. The formula doesn't know what an options market is and even less so what the meaning of inverting the formula and implying volatility from the option market price could possibly be.

Derivatives trading and the real meaning of BSM

I wish now to introduce a difficult thought and that is that the constant volatility of BSM, which is not constant in time or space but is constant in the world of BSM (by which I mean that it is a concept, that it reaches until the end of times, that it is equal to the meaning of trading, etc.), will change because the world of BSM changes, *and that the only way this world can change is through the trading of the derivative which was not supposed to trade*. This change of volatility does not mean stochastic volatility (or a process of change taking place in time), even though the common response to the phenomenon of derivatives trading has always been to make volatility stochastic. In fact, I wish to argue that this change (which is not supposed to take place because the world of trading is still a concept that goes until the end of times without change) can only take place in a structure where everything changes while nothing changes. This structure is the regime-switching structure. I wish to argue that the only way that indefinite volatility can change (indefinite because it means trading conceptually, up to and including the end of times) is through a structure, the regime-switching structure, which only looks like a simplified or a discretized stochastic volatility process from the outside, but whose main virtue, really and intrinsically, is its auto-similarity or its capacity of rescaling.[7] The bank account and trading, or the two activities that are not contained in finite time because they are the presupposition of time, cannot change through a temporal process happening within time. They change through a change of the world or a superposition of temporalities, and this can only be made possible by the regimes. This is our thesis. Something perpetual cannot change unless a rescaling of time and of the world takes place. Because of the perpetuity and the universality of trading (everything trades), it is the differential of the moment that counts, not the integral, hence the regimes as the unique solution.

It is wrongly thought that options trade because volatility is stochastic and option trading is a bet on volatility. At best, a stochastic volatility process is one

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way of explaining, in an arbitrage-free way, options prices that exhibit BSM volatility smiles. Or the common thought is that in a BSM world with constant (i.e. nonstochastic) volatility, options are redundant, hence cannot trade, so if they trade, then volatility must be stochastic and not constant. At best, this is the converse implication. In reality, stochastic volatility does not imply and does not even explain option trading. A snapshot of options prices is taken and given that BSM cannot explain them, it is thought that a stochastic volatility process can. But explaining the snapshot is not explaining the trading. Happy as you may be that your stochastic volatility process has explained the vanilla options volatility smile, this will not induce you to trading them. An explanation is a cause of rest and satisfaction, not of activity. Stochastic volatility is not the cause of option trading. Volatility is stochastic independently of option trading and of the whole world of the option market-maker. Volatility is stochastic in the world of the econometrician. *What causes option trading is BSM and perfect replication of the options by the market-maker*. It is because options market-makers know that volatility is constant in their BSM world, and know with certainty how to replicate and manufacture the options, that they write them and make their market. (This is not knowledge in the sense of epistemology, as if market-makers would be wrong to know that volatility is constant when in reality it is not; they are not wrong – their knowledge is not knowledge of facts or of propositional truths, it is a knowhow; it is knowing *how*, not knowing *that*.)

We cannot make anything (and least of all, make a market) when we only have uncertainty and no certainty. Options trade because of the BSM world and because of volatility being constant in that world (i.e. conceptually constant, constant as in the meaning of trading). Volatility is constant (in that world) and should remain so, not until it becomes stochastic, but until the option trades. This "until" is not inscribed in time, hence it is not inscribed in epistemological knowledge. Rather, the first attempt at making the indefinite and perpetual volatility definite – its first "definition," we may say – is the option price. Volatility and Brownian motion are deduced from the infinite causes of Bachelier or from the continuous trading in fractional size of Vovk, and, as such, volatility is constant because no variation is open to it; it is an indefinite and perpetual concept. What BSM have achieved is to complete this deduction. Indeed, one may say that the concept of trading of a certain underlying stock was not yet complete with the Brownian motion of the price of that stock, for volatility, which was the result, was not (yet) tradable. What completes the concept of trading is the association of Brownian motion with self-financing dynamic trading in fractional size. That is to say, the completion of the concept of trading of the underlying stock S is $B + \Delta S$, or the writing of the derivative. BSM has made it so that the option premium, or the initial cost of the self-financing dynamic replication strategy of the option, *is the ultimate concept of the trading of the underlying stock*. And now the important thing to note is that this option premium can become a price offered by the market-maker *only insofar as the market-maker is certain of his replication formula and certain that volatility is constant in his world (in his formula)*. The option premium can become a price, that is to say, it can start moving randomly in its turn and start being affected by the infinite causes of Bachelier in ways that will drive it away from the prescription of the formula, because of the very prescription of the formula. Implied volatility can start becoming stochastic (and this is different from saying that volatility becomes stochastic), that is to say the options market can start, because volatility is (and must remain) constant in the formula. This is the paradoxical association of change and no change that the perpetuity of trading implies, and which generates all the difficulty of the smile problem. Insofar as debt is converted into equity and the perpetuity of the bank account becomes geared to the perpetuity of trading, or insofar as the end of the world of debt is converted into the end of the world of trading, the bank which was not supposed to fail (being a presupposition of time) but which fails (because of the introduction of finite time) is transformed into the derivative, which is not supposed to trade (because it is the concept and the end of the world of trading) but which trades (because of the introduction of the market-maker who is so very sure of the concept of the underlying stock trading that he writes the derivative as a result, and because writing the derivative can only mean its trading).

The reason why the derivative is not supposed to trade but represents, instead, the end of the world of trading, is that it is valued and that *its value is deduced from nothing else other than the concept of trading, or the concept of price, which is the absolute opposite of value*

The reason why the derivative is not supposed to trade but represents, instead, the end of the world of trading, is that it is valued and that *its value is deduced from nothing else other than the concept of trading, or the concept of price, which is the absolute opposite of value*. The underlying stock trades and its price is random at any scale because of the absolute failure of the notion of value. Stock prices rise in the stock exchange for no other reason than people buying the stock and expecting the price to rise, and prices fall for the opposite reason. This is Keynes's famous beauty contest. This is speculation in Bachelier, in which the stock exchange has nothing other than itself to speculatively look at and speculatively reflect. Brownian motion and volatility are the result of this efficient market in which there is only price and no value – in which there is nothing to look at as a guide for price or as an anticipation (a predictor) of its motion. And now the amazing thing is that, with volatility thus becoming the only ground and only fundamental value of the market, or the only thing that the market means, BSM are able impeccably to deduce the option value. As sure as there is no value and there is only price, that is to say, as sure as there is only the volatility of price, there is now the *value* of the option. The option valuation is not only impeccable; it is unassailable and there is no way, indeed, that it could be criticized. It is itself engendered by nothing else but the absolute criticism of value, which is the constant and perpetual volatility of price. Gods looking at the stock exchange and understanding its fundamental concept (or value), which is volatility, are thus able to value the options. By the same token, this means that the option value belongs on the conceptual level of the market and cannot mix, in any way, with its object level. Maybe gods could trade the option; maybe the option could be traded; however, if it must be traded, it could trade anywhere except in the same trading pit as the underlying stock. This is why we are saying that it cannot be traded, and that trading it is the end of the world of trading.

The necessity of the regime-switching model

That the option cannot be traded (because it is equal to the very concept of trading) is at the same time the very reason why it is traded (because of the certainty of the market-maker writing it and the certainty of the concept of trading and therefore the certainty of the end of the world of trading). The smile problem is the exact consequence of this twist and tension. We use the BSM implied volatility when we are no longer supposed to. Trying to include the market-maker in the explanation can only mean that trading the stock is trading the derivative is trading the derivative written on the derivative, etc. So, the only true situation is one always comprised between situations. Continuous trading in fractional size implies Brownian motion from arguments that have nothing to do with probability but only with the nature of money and the capital process (the argument, according to Vovk, that no one should get infinitely rich). The mistake is then to think that the problem becomes one of econometrics and that the next move is to analyze time series of the underlying price and provide temporal criticisms of Brownian motion to the effect that volatility is in fact stochastic, or the paths are not continuous, or the distribution must be scalable (Mandelbrot and Taleb), etc. By posing the problem in this fashion, we bring the problem to a stop. It is no longer a trading problem. In this view, trading was only a pretext, a generator of underlying prices, and next we content ourselves with the study of probability distributions of the underlying price, which can only extend in so many stages. There are jumps and there is stochastic volatility, and the jump sizes and frequencies are themselves stochastic and the volatility of volatility is stochastic in turn, etc. The thought is that there is some ultimate process to be discovered, which keeps evolving constantly, of course, and will never become stationary, a process whose parameters will keep indefinitely becoming stochastic, of course, however, a process that is fixed at any one time. This is wrong. The only way to advance the problem after Brownian motion should be, on the contrary, that the derivative whose premium, we said, was even better and more complete than the volatility of Brownian motion at expressing the concept of trading of the underlying, must trade in turn, and that its impeccable and unassailable value must become a price in turn. From the beginning, the lesson from Bachelier should have been the trading of the derivative in infinite combinations and the return to the trading floor of anything we may have deduced above the trading floor.[8] From the beginning, it should have been suspicious that something starting from the trading floor should find an exit in the study of time series, and find a stop because the study of time series is finite, when trading is infinite. From the start, the models should have been *market models* (and BSM should have been understood first as a market model) in which both the underlying and the derivative deviate away from something Lorenzo Bergomi rightly calls a *pricing formula* (and no longer a model).[9]

Options become tradable and bundled in the volatility index (VIX). Futures on VIX become tradable in turn, as well as options written on these futures, and so on and so forth. When it is understood that anything trading generates the premium of the option written on it as the conceptualization of its trading and the completion of this concept, and when it is understood that this premium (and conceptualization) is soon to join the trading floor in its turn (thus showing what is most amazing in the concept of trading; that it cannot really be conceptualized unless the completion of its concept becomes equal to it again), this infinite chain, which by construction cannot be brought to an end or to completion, forces us to look for the differential rather than the integral structure. While econometrics and the study of time series are extensive, the trading pit is intensive.

Indeed, what needs to be added, all the time, or the way the structure is supposed to accrete, all the time, is a strange kind of addition or accretion. At any point, a certain number of things are being traded and the concept of their trading is expressed by constant (i.e. conceptual) volatilities. If a number of independent stocks are being traded, then what we said above about the volatility of a single stock price being constant because it is just the conceptualization of its trading, generalizes to several constant volatilities and correlations – constant because they are just symbols, which won't become quantified until options written on those stocks, either with a single stock as an underlying or a basket thereof, admit of quantified values (i.e. prices – for price is the only quantity in our world; volatility of time series is not the right way to quantify). This is just a horizontal generalization of the case of the single stock, and it poses no problem. However, the number of things that are being traded in our case are not independent assets, but derivatives written one upon the other, and ultimately upon a single underlying stock. The volatility of the underlying stock becomes quantified when a single option written on it becomes traded; however, the volatility

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of volatility, or the size and frequency of the jumps that we need to introduce, become quantified in their turn when the whole option smile becomes traded (indeed, its shape depends on the magnitude of stochastic volatility and the magnitude of jumps), or better, when second-generation derivatives, which are precisely directly sensitive to stochastic volatility and to jumps, for instance VIX options, or cliquet options, or barrier options, become traded. The volatilities that we said were constant, just because they expressed the concept of trading of instruments written so far, are in this case "generalized volatilities," typically the parameters of the regime-switching model (Brownian volatilities and jumps within the regimes, jumps ruling the transitions between the regimes, etc.) stopped at a certain level (i.e. at a certain number of regimes).

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These volatilities are constant and certain; they are as constant and certain as the concept of trading of the instruments written so far; however, the main interesting thing is that the certainty in question serves one sole purpose, that of the market-maker replicating, therefore writing, therefore trading, derivative instruments of the next generation, written on the instruments written so far. What is interesting is the addition, through the meaning of trading of instruments of lower generation, of the trading of instruments of higher generation – an intensive addition. We know what the extensive result is. The extensive result is a picture in which the infinite hierarchy of derivatives are all being traded at the same time. However, the extensive result, even when driven to its infinite limit, misses the intensive infinity, which expresses itself in the *market-maker* (i.e. in the *meaning* of the pit). What we need to express is not the extensive end result but precisely the hinge, which operates at every level (to infinity), and which precisely introduces the dynamics of replication/writing/trading of the *next* derivative. The situation, we said, is always the next situation, and the pit and the market (intensively understood) are always the trading of the *next* derivative. Reality is recalibration. To introduce the pit is to introduce the market-maker, therefore it is to introduce the hinge. A picture brought to its extensive end result misses the market-maker. The market-maker, or the pit, or the hinge, occurs when we capture the fact that constant and conceptual volatility *is precisely the reason why it will no longer be constant, because it drives the market-maker to writing and making the market of the next derivative*. This, we submit, is made possible by the regime-switching model because in its extensive structure (an identified and fixed number of regimes) it hides the intensive addition of structure, due to self-similarity. The regime structure is at the same time finite (i.e. prone to change and prone to recalibration) and already all there, intensively, to infinity.

Generalizing Brownian motion in time (i.e. by extracting it from the trading pit and by finding stochastic volatility after constant volatility, or jumps after the continuous path) is not a *truthful* generalization of BSM. The generalization must take place in the pit, not in time. It should be understood that BSM is a conceptual model, not a time series model, where volatility is not constant in time, but is constant before time, as the concept of trading in the pit. BSM should be generalized by making the infinite and indefinite universe finite, in other words, by turning it into a world. Generalization amounts to understanding the relation between what is formal (infinite and indefinite) and what is material in the BSM model and argument, and by maintaining *this* relation. BSM is not a formal theory in the sense that empirical and material reality is being modeled, and that the formal register has to disappear after serving its purpose. (At every stage of the generalization, or of the recalibration, the formal register exists.) First, one has to understand that B, the bank, and ΔS, the trading, are formal and for this reason perpetual, and then one has to introduce finite time in this indefinite picture, by recognizing the credit of the bank and by recognizing the trading of the derivative. These are the two events that break the indefinite temporal and spatial symmetry of the formal world. These two events signal the end of the formal world. They amount to recognizing a conversion, and they are expressed by two modes of conversion, which are inverse one of the other, one going to the left, towards B, the bank, and one to the right, towards ΔS, the trading pit. After the conversion, the horizon can become perpetual again. The end of the world of the bank is re-immersed in time through the bail-in event and the bail-in structure in the perpetual debt, and the end of the world of trading is re-immersed in the trading pit through the perpetual chain of derivative writing and trading (something we may call *trade-in*).

$B + \Delta S$ is the conversion. It is a formal and even a philosophical operator; for this reason, its generalization should proceed formally, and then become material in a surprising way, through the CoCo bond, going to the left, and the vanilla convertible, going to the right. Regimes of stochastic credit, to the left, and regimes of stochastic volatility, to the right. The only way to keep the formal argument (i.e. to keep the constant and indefinite and qualitative credit and volatility) while embedding definite and finite time in it (i.e. while embedding the world and the change of the world, that is to say, **the discrete rescaling of volatility and credit, not the turning of volatility and credit into variables that change continuously in time**); the only way to keep the formal argument, while embedding in it the credit of the bank (which was supposed to have none) and embedding in it the trading of the derivative (which was supposed never to trade), is through the regimes: constant, while at the same time embedding change. The regimes are constant as the qualitative argument requires, and for this reason they accord with the perpetuity of the instruments. It is because the CoCo bond is perpetual and the convertible bond is perpetual (in this case, understood as the right side of the conversion, as the perpetual writing and trading and re-immersion of the derivative) that the regimes are the answer. They generalize BSM, not in the sense of featuring stochastic volatility and jumps (something they definitely do), but in featuring the change of BSM, a change happening by the end of the world, by the twist of the constant volatility, which is implied volatility. Volatility doesn't change in BSM because it becomes stochastic; it changes because it becomes implied volatility. It doesn't (quantitatively) change; it is ex-changed.[10] This is what must be generalized, the *truthful generalization*.

The conversion $B + \Delta S$ is already conducive to the trading of the next derivative, that is to say, to the trading of the full chain of derivatives. If, as we speculate,

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$B + \Delta S$ is the mutation from the time of credit (where the only expected event is the end of the world and hyperinflation) to the time of trading, or to the time of speculation in the sense of Bachelier and of the stock exchange looking at itself, then, in this qualitative mutation, neither Bachelier nor BSM (which is its financial completion and the shifting of Bachelier from the world of probability to the world of arbitrage) can understand volatility in time, as a time process, or a parameter that will change in time. Volatility is qualitative, in this instance: a concept.

BSM is truly an inaugural formula and model (so inaugural that it will inaugurate the trading of derivatives, unbeknownst even to Black, Scholes and Merton), and it would have indeed surprised us a lot if Black, Scholes and Merton had started with non-constant volatility. We have to understand the formal register here, and keep this understanding. Black, Scholes and Merton could not but consider a constant (conceptual) volatility, because, unbeknownst to them, the next thing in their formula was going to be the writing and the trading of the derivative (i.e. implied volatility, or the breaking of the symmetry and indifference and in-definition of the concept of volatility through the trading

Only the regimes can be fixed in number and at the same time embed the virtual fact that the number can be more, as certain regimes are hidden and not yet apparent and distinguished behind their superposition with the initial ones

of the derivative). At every stage, the volatility, or later the volatilities, have to be constant (homogenous, independent of time and space, because the concept is homogenous, and its change or rescaling will happen via the end of the world, not in time or in space), and the only change that this constant and homogenous and conceptual picture has to be able to represent, is the virtual change through the trading of the derivative. In other words, the only sign of change is the possibility of rescaling, and of self-similarity of the structure with the next structure.

Only the regimes can be fixed in number and at the same time embed the virtual fact that the number can be more, as certain regimes are hidden and not yet apparent and distinguished behind their superposition with the initial ones. This, in itself, is the proof that the structure has to be discrete. The imperative in the generalization is to respect what BSM is accomplishing: the change of the world, the trading of the derivative, which wasn't supposed to trade; it is to perpetuate the real argument, which was completely overlooked and which is that BSM is totally a formal argument, soon to be materialized by writing and by trading, not by an exit in time. From indefinite and formal, the world becomes finite and material (i.e. material time makes an entry – the material time of the end of the world by hyperinflation, or the material time of the material exchange of the actual derivative contract, which can no longer be viewed as an algorithmic schedule, but has to turn itself to trading as one material piece, as the material support of the written schedule), thanks to the structure of bail-in, which makes perpetual debt possible again, and thanks to the structure of trade-in, or the exchange in which underlying stock and derivative trade in concert. The formal and perpetual world steps into the material world through the unsuspected entry: the CoCo bond and the convertible bond, which produce the discrete regimes of credit and volatility, respectively – a material world of prices and trading, not a material world of time exit and statistics.

ENDNOTES

1. Ferguson, N. 2008. *The Ascent of Money: A Financial History of the World*. London: Allen Lane.
2. A quote attributed to Frederic Jameson, which he himself attributes to someone else.
3. Bachelier, L. 2006. *Theory of Speculation: The Origins of Modern Finance*. Princeton, NJ: Princeton University Press (translated and with commentary by Mark Davis and Alison Etheridge).
4. Vovk, V. 2012. Continuous-time trading and the emergence of probability. *Finance and Stochastics* 16(4), 561–609.
5. Bary, A. 2020. Convertible bond issuance is booming. Here's where to find opportunities. *Barron's magazine*, May 22.
6. Neftci, S. N. 2000. *An Introduction to the Mathematics of Financial Derivatives*, 2nd edn. London: Academic Press.
7. Technically, the regime-switching structure is such that the volatility and the hazard rate are constant in each regime and depend on that regime. Switches between regimes occur through Poisson processes that also trigger a jump in the underlying asset price. Regime switches thus correlate jumps of the volatility and the hazard rate with jumps of the underlying asset price. Auto-similarity is the fact that a given subset of regimes can be considered as a single "big" regime, and switching in the larger set can be viewed as occurring between such big regimes. A regime-switching structure is thus indistinguishable from its own stochasticization, and it is never determined whether adding an *n*th regime, say of volatility, is just expanding the current stochastic volatility model or making the volatility of volatility stochastic in turn.
8. As soon as we turn the first page of Bachelier's book, we read: "There are two sorts of forward-dated transactions: forwards, options. These transactions can be combined ad infinitum, especially as one often treats several kinds of option."
9. Bergomi, L. 2016. *Stochastic Volatility Modeling*. Boca Raton, FL: CRC Press.
10. Hélyette Geman has expressed this nicely by writing: "The economy of BSM is risky by definition because it amounts to exchanging volatility. The fact that this risk should be materialized by a single little number σ makes it palpable and immediate for everybody, and so nobody can use the complexity of the model as an excuse." [From Bachelier to Black–Scholes–Merton, *Bulletin français d'actuariat* 1(2), 1997]. Geman formulates perfectly the smile problem or paradox: σ is both constant ("a single little number") and non-constant (it is ex-changed).